Tikrit university
College of Engineering
Mechanical Engineering Department

Lectures on Engineering Analysis

Chapter 1
Complex analysis

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Complex Analysis

The definition of the imaginary number: The square root of a negative real number is not a real number. Thus, we introduce imaginary numbers :

$$i = \sqrt{-1}$$

A complex number z is a number of the form

$$Z = x + yi$$

Real part (x) — **Imaginary part** (yi)

BASIC ALGEBRAIC PROPERTIES

Various properties of addition and multiplication of complex numbers are the same as for real numbers.

Two complex numbers

are added (or subtracted) by adding (or subtracting) real number parts and imaginary coefficients, respectively.

$x_1 + iy_1$ and $x_2 + iy_2$

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + (iy_1 + iy_2) = z_1 + z_2$$

$$(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + (iy_1 - iy_2) = z_1 - z_2$$

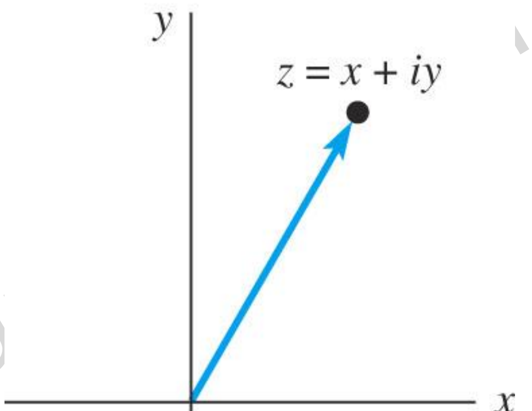
Example

$$(2 + 4i) + (5 + 3i)$$
$$= (2 + 5) + (4 + 3)i = 7 + 7i$$

Multiplication and Division

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

$$\frac{z_1}{z_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$



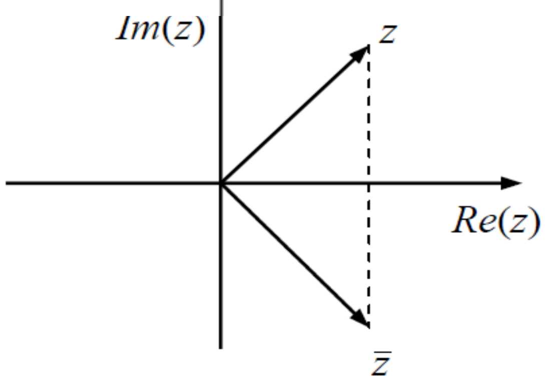
Complex Plane

The coordinate plane illustrated in Figure is called the **complex plane**. The horizontal or *x*-axis is called the **real axis**. The vertical or *y*-axis is called the **imaginary axis**.

The complex conjugate

The complex conjugate of $z = x + iy$ is written $\bar{z}$ and is defined by

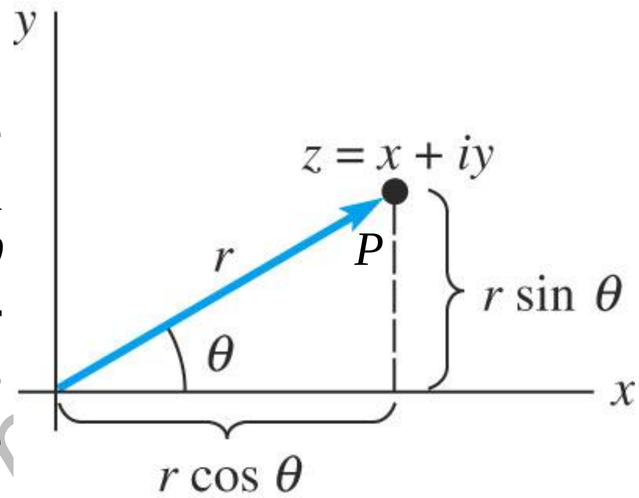
$$\bar{z} = x - iy$$



The modulus or absolute value of $z = x + iy$, denoted by $|z|$, is the real number

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$$

Polar Form of a Complex Number



In the polar coordinate system, consists of point *O* called the **pole** and the horizontal half-line emanating from the pole called the **polar axis**. If *r* is a directed distance from the pole to *P* and *θ* is an angle of inclination (in radians) measured from the polar axis to the line *OP*, is positive when measured counterclockwise and negative when measured clockwise. then the point can be described by the ordered pair (*r*, *θ*), called the polar coordinates of *P*.

$$\tan\theta = \frac{y}{x}$$

$$\arg z = \theta, \quad \text{where} \quad \cos\theta = \frac{x}{|z|}, \quad \sin\theta = \frac{y}{|z|}$$

The use of polar coordinates (r, θ) : (x, y) = (r cos θ, r sin θ) gives the polar form. For complex number *z* = *x* + *yi* is

$$x = r\cos\theta, \quad y = r\sin\theta$$

We see that then $z = x + iy$ takes the so-called **polar form**

$$z = r(\cos\theta + i\sin\theta)$$

Example Express $1-\sqrt{3}i$ in polar form.

Solution

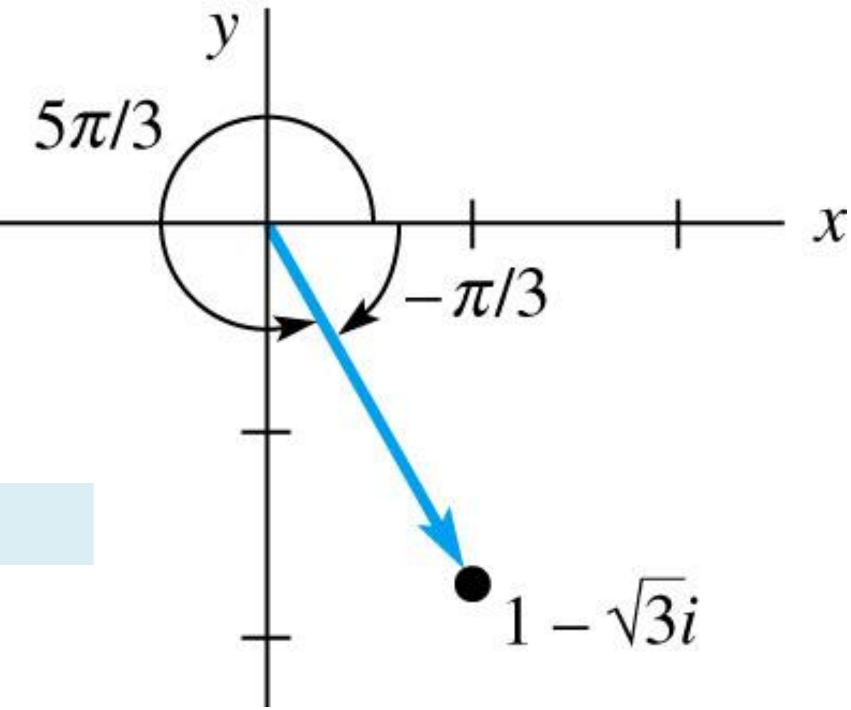
See Fig that the point lies in the fourth quarter.

$$z = r(\cos\theta + i\sin\theta) \qquad |z| = \sqrt{x^2 + y^2}$$

$$r = |z| = |1-\sqrt{3}i| = \sqrt{1+3} = 2$$

$$\tan\theta = \frac{-\sqrt{3}}{1}, \theta = \arg(z) = \frac{5\pi}{3}$$

$$z = 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]$$



Multiplication and Division

Suppose $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

for $z_2 \neq 0$,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$

From the addition formulas from trigonometry,

$$z_1z_2 = r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

$$|z_1z_2| = |z_1||z_2|, \quad \left|\frac{z_1}{z_2}\right|, = \frac{|z_1|}{|z_2|}$$

$$\arg(z_1z_2) = \arg z_1 + \arg z_2, \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

Functions of a complex variable:

All elementary functions of real variables may be extended into the complex plane. Now in complex, S is a set of *complex* numbers. And a **function** f defined on S is a rule that assigns to every z in S a complex number w, called the *value* of f at z. We write

Function of a complex variable : $w = f(z).$

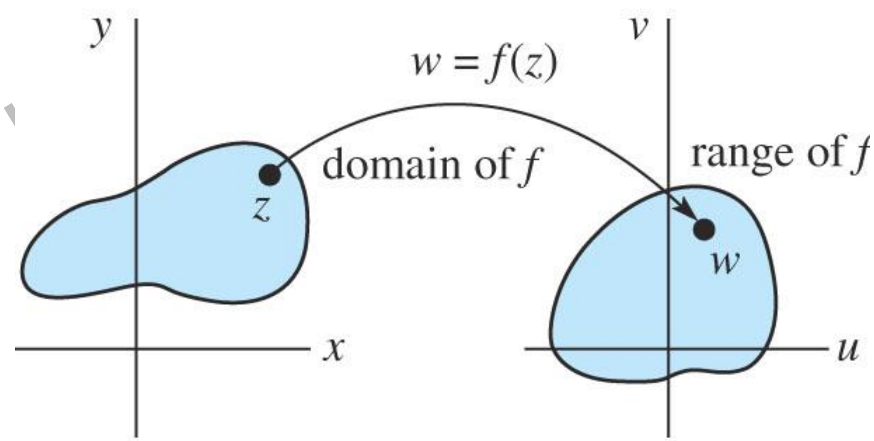
Therefore, complex function can be resolved into its *real part* and *imaginary part*:

$$z = x + iy, \quad w = u + iv$$

w is complex, and *u* and *v* are the real and imaginary parts, respectively.

$$w = f(z) = u(x,y) + iv(x,y)$$

u is a real function of *x* and *y*, and so does *v*.



(a) z-plane (b) w-plane

Example

Let $w = f(z) = z^2 + 3z$ find $u\ and\ v\ and\ calculate\ the\ value\ of\ f\ at\ z = 1 + 3i$

Solution

$$f(z) = x^2 + 2xyi - y^2 + 3x + 3yi$$

$$z = x + yi$$

$u = x^2 - y^2 + 3x$ and $v = 2xy + 3y$

$$f(1 + 3i) = (1 + 3i)^2 + 3(1 + 3i)$$
$$= 1 - 9 + 6i + 3 + 9i$$
$$= -5 + 15i$$

This shows that $u(1,3) = -5\ and\ v(1,3) = 15$.

Check this by using the expression for $u\ and\ v$

Example

Consider $\boldsymbol{f(z) = z^2 + iz}$, and express it in terms of real and imaginary parts.
This is of the form $w = u + iv$

$$f(z) = x^2 + 2xyi - y^2 + ix - y$$

$u = x^2 - y^2 - y$ and $v = x + 2xy.$

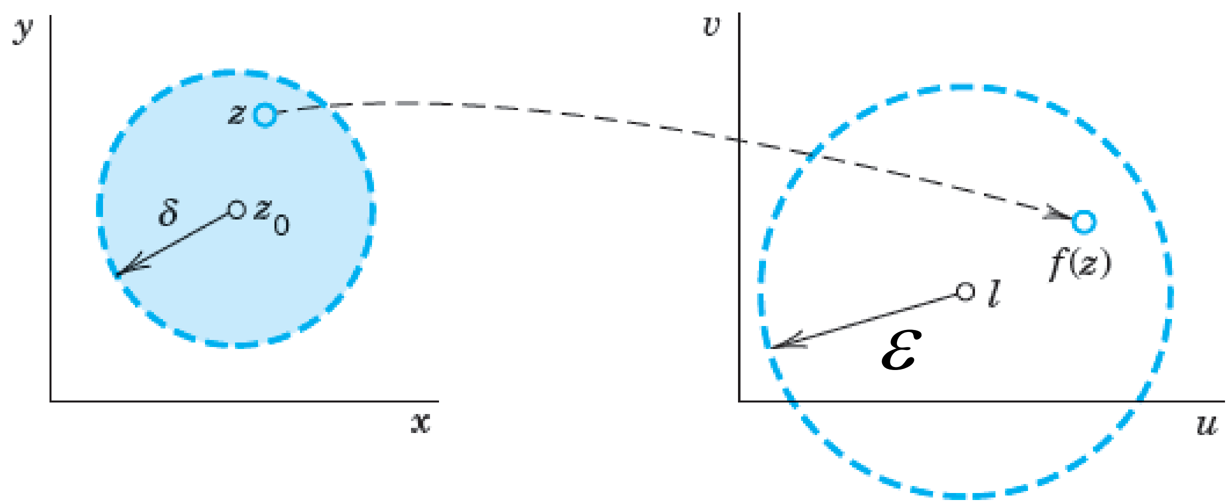
Limit of a Function

Suppose the function f is defined in some neighborhood of z_0, except possibly at z_0 itself. Then f is said to possess a **limit** l *as z approaches a* z_0, written

$$\lim_{z \to z_0} f(z) = L$$

if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(z) - L| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$



z may approach *Z0* ***from any direction*** in the complex plane. This will be quite essential in what follows.

If a limit exists, it is unique.

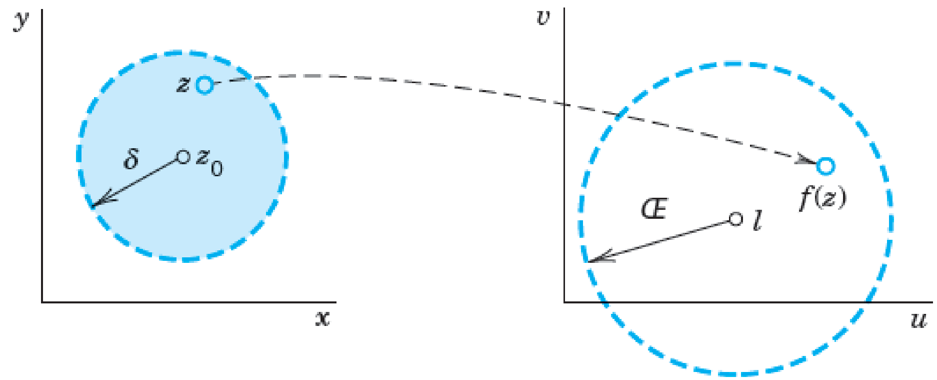
Limit of Sum, Product, Quotient

Suppose $\lim_{z\to z_0} f(z) = L_1$ and $\lim_{z\to z_0} g(z) = L_2$.

(i) $$\lim_{z\to z_0}[f(z)+g(z)] = L_1 + L_2$$

(ii) $$\lim_{z\to z_0} f(z)g(z) = L_1 L_2$$

(iii) $$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0$$



Continuous Function

A function f is continuous at a point z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0)$$

A function f is said to be *continuous on set S* if it is continuous at each point of S. If a function is not continuous at a point, then it is said to be singular at the point.

Example

Consider the function $f(z) = z^2 - iz + 2$. In order to determine if f is continuous at, say, the point $z_0 = 1 - i$, we must find $\lim_{z \to z_0} f(z)$ and $f(z_0)$, then check to see whether these two complex values are equal.

$$\lim_{z \to z_0} f(z) = \lim_{z \to 1-i} \left(z^2 - iz + 2\right) = (1-i)^2 - i(1-i) + 2 = 1 - 3i.$$

Furthermore, for $z_0 = 1 - i$ we have:

$$f(z_0) = f(1-i) = (1-i)^2 - i(1-i) + 2 = 1 - 3i.$$

Since $\lim_{z \to z_0} f(z) = f(z_0)$, we conclude that $f(z) = z^2 - iz + 2$ is continuous at the point $z_0 = 1 - i$.

Derivative

Suppose the complex function f is defined in a neighborhood of a point z_0. The derivative of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$ provided this limit exists.

- If the limit in above equation exists, f is said to be differentiable at z_0. Also, *if f is differentiable at z_0, then f is continuous at z_0.*

Rules of differentiation

- **Constant Rules:** $\frac{d}{dz} c = 0\,, \quad \frac{d}{dz} cf(z) = cf'(z)$
- **Sum Rules:** $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$
- **Product Rule:** $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + g(z)f'(z)$
- **Quotient Rule:** $\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$
- **Chain Rule:** $\frac{d}{dz} f(g(z)) = f'(g(z))g'(z)$
- **Usual rule** $\frac{d}{dz} z^n = nz^{n-1}\,, \quad n$ an integer

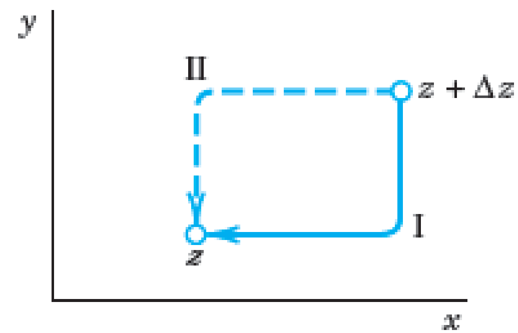
Example

Differentiate $(a)\, f(z) = 3z^4 - 5z^3 + 2z, (b)\, f(z) = \frac{z^2}{4z+1}.$

Solution $(a)\, f'(z) = 12z^3 - 15z^2 + 2$

$$(b)\, f'(z) = \frac{(4z+1)2z - z^2 4}{(4z+1)^2} = \frac{4z^2 + 2z}{(4z+1)^2}$$

Example



Show that $f(z) = x + 4iy$ is nowhere differentiable.

Solution

With $\Delta z = \Delta x + i\Delta y$, we have

And so $f(z + \Delta z) - f(z)$

$$= (x + \Delta x) + 4i(y + \Delta y) - x - 4iy$$

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}$$

Now if we let $\Delta z \to 0$ along a line parallel to the x-axis then $\Delta y = 0$ and the value of above results is 1. On the other hand, if we let $\Delta z \to 0$ along a line parallel to the y-axis then $\Delta x = 0$ and the value of above results is 4. Therefore $f(z)$ is not differentiable at any point z.

Analyticity at a Point

A complex function $w = f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

Cauchy-Riemann Equations

Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$.

Then at z the first-order partial derivatives of u and v exists and satisfy the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (1)$$

Proof

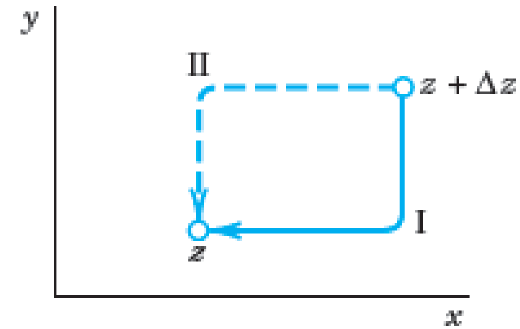
Since $f'(z)$ exists, we know that

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \qquad (2)$$

By writing $f(z) = u(x, y) + iv(x, y)$, and $\Delta z = \Delta x + i\Delta y$, form (2)

$$f'(z) = \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

Since the limit exists, Δz can approach zero from any direction. In particular, if $\Delta z \to 0$ horizontally, $\Delta y = 0$ then $\Delta z = \Delta x$ and (3) becomes



$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \qquad (4)$$

By the definition, the limits in (4) are the first partial derivatives of u and v w.r.t. x. Thus

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad f'(z) = u_x + i v_x. \qquad (5)$$

Similarity, if we choose path *II* in above figure, we let $\Delta z \to 0$ first and then $\Delta x \to 0$. After Δx *is zero*, $\Delta z = i\Delta y$, so that from *(3)* we now obtain

Now if $\Delta z \to 0$ vertically, then $\Delta z = i\Delta y$ and (3) becomes

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta y \to 0} \frac{iv(x, y+\Delta y) - iv(x, y)}{i\Delta y}$$

which is the same as
$$f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad f'(z) = -i u_y + v_y.$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Therefore, these two expressions must be equivalent. Equating real and imaginary parts, we have that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy–Riemann Equations

If two real-valued continuous functions $u(x, y)$ and $v(x, y)$ of two real variables x and y have continuous first partial derivatives that satisfy the Cauchy–Riemann equations in some domain D, then the complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

- We mention that, if we use the polar form $z = r(\cos\theta + i\sin\theta)$ and set $f(z) = u(r,\theta) + iv(r,\theta)$, then the **Cauchy–Riemann equations** are

$$u_r = \frac{1}{r}v_{\theta}, \qquad v_r = -\frac{1}{r}u_{\theta}$$

Example

Cauchy–Riemann Equations. Exponential Function

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Is $f(z) = u(x, y) + iv(x, y) = e^x(\cos y + i \sin y)$ analytic?

Solution. We have $u = e^x \cos y$, $v = e^x \sin y$ and by differentiation

$$u_x = e^x \cos y, \qquad v_y = e^x \cos y$$

$$u_y = -e^x \sin y, \qquad v_x = e^x \sin y.$$

We see that the Cauchy–Riemann equations are satisfied and conclude that $f(z)$ is analytic for all z. ($f(z)$ will be the complex analog of e^x known from calculus.)

Example : is $f(z) = (3x^2 - 3xy + x - 3y^2 + 2y) + i(-x^2 - 6xy - 2x + y^2 + y)$ analytic or not in complex plane.

Solution $u = 3x^2 - 3xy + x - 3y^2 + 2y$ $\quad v = (-x^2 - 6xy - 2x + y^2 + y$

$$\frac{\partial u}{\partial x} = 6x - 3y + 1 \qquad \frac{\partial v}{\partial y} = -6x + 2y + 1$$

$$\frac{\partial u}{\partial y} = -3x - 6y + 2 \qquad \frac{\partial v}{\partial x} = -2x - 6y - 2$$

The equations do not hold in the neighborhood of any point and thus $f(z)$ is **not analytic** at any point.

Example : For the function $f(z) = e^x (\cos(y) - i\sin(y))$, determine whether or not it is analytic in the complex plane. is analytic everywhere.

Solution :

$$u = e^x \cos(y) \qquad v = -e^x \sin(y)$$

Hence

$$\frac{\partial u}{\partial x} = e^x \cos(y) \qquad \frac{\partial v}{\partial y} = -e^x \cos(y)$$

Then

$$\frac{\partial u}{\partial y} = -e^x \sin(y) \qquad \frac{\partial v}{\partial x} = -e^x \sin(y)$$

Cauchy Riemann equations are not satised at any point and hence f is not analytic at any point

Laplace's Equation. Harmonic Functions

A real-valued function $\phi(x, y)$ that has continuous second-order partial derivatives in a domain D and satisfies Laplace's equation is said to be **harmonic** in D.

If $f(z) = u(x, y) + i\, v(x, y)$ is analytic in a domain D, then u and v both satisfy in D the Laplace's equation:

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

(∇^2 read "nabla squared") *and*

$$\nabla^2 v = v_{xx} + v_{yy} = 0,$$

in D and have continuous second partial derivatives in D.

Harmonic Conjugate;

If *u* and *v* are harmonic in *D*, and *u(x,y)+iv(x,y)* is an analytic function in D, then *u* and *v* are called the conjugate harmonic function of each other.

Such a function *v* is called a *harmonic conjugate* of *u*.

Example 4

(a) Verify $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane.
(b) Find the conjugate harmonic function of u.

Solution

$$(a)\ \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \frac{\partial^2 u}{\partial x^2} = 6x, \frac{\partial u}{\partial y} = -6xy - 5, \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$$(b)\quad \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 3x^2 - 3y^2 \ \text{ and } \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6\text{xy} + 5$$

Integrating the first one, $v(x, y) = 3x^2 y - y^3 + h(x)$

and $\frac{\partial v}{\partial x} = 6xy + h'(x), h'(x) = 5, h(x) = 5x + C$

Thus $v(x,y) = 3x^2 y - y^3 + 5x + C$

Example 4

- (a) Verify $u(x, y) = 2+3x - y + x^2 - y^2 - 4xy$ is harmonic in the entire complex plane.
  (b) Find the conjugate harmonic function of u satisfying $v(0; 0) = 0$.

Solution The Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

$$\frac{\partial u}{\partial x} = 3 + 2x - 4y, \qquad \frac{\partial u}{\partial y} = -1 - 2y - 4x$$

and

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 u}{\partial y^2} = -2, \qquad \nabla^2 u = 0.$$

We can obtain a harmonic conjugate by using the Cauchy Riemann equations.

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 1 + 2y + 4x.$$

Partially integrating with respect to x gives

$$v(x, y) = x + 2xy + 2x^2 + g(y)$$

where $g(y)$ is any differentiable function of y. Partially differentiating this expression and using the other Cauchy Riemann equation gives

$$\frac{\partial v}{\partial y} = 2x + g'(y) = \frac{\partial u}{\partial x} = 3 + 2x - 4y.$$

Hence

$$g'(y) = 3 - 4y, \quad g(y) = 3y - 2y^2 + C$$

where C is a constant. To satisfy $v(0, 0) = 0$ we need $v(0, 0) = g(0) = C = 0$ and thus

$$v(x, y) = x + 2xy + 2x^2 + 3y - 2y^2.$$

Exponential Function

The complex exponential function is one of the most important analytic functions

$$e^z = e^x (\cos y + i \sin y)$$

If $z = 3 + 4i$ then

$$e^z = e^3 (\cos 4 + i \sin 4)$$

For real z = x, imaginary part y = 0

$$e^z = e^x (\cos y + i \sin y) \quad (\cos y \to 1,\ \sin y \to 0) \qquad e^z = e^x$$

e^z is analytic for all z

Example 1

Evaluate $e^{1.7+4.2i}$.

Solution

$$e^{1.7+4.2i} = e^{1.7}(\cos 4.2 + i \sin 4.2)$$

$$= -2.6873 - 4.7710i$$

General rule of the exponential functions the

The derivative of the exponential function is:

$$(e^z)' = e^z.$$

Recall that the function $f(x) = e^x$ has the property

$$e^a \times e^b = e^{(a+b)}$$

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1)\, e^{x_2}(\cos y_2 + i\sin y_2).$$

$$e^{z_1}e^{z_2} = e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)] = e^{z_1+z_2}$$

Since $z = x + iy$

$$e^z = e^{(x+iy)} = e^x\, e^{iy}$$

For pure imaginary complex number where $z = iy$

$$e^{iy} = \cos y + i\sin y$$

Euler's formula is

Polar From of a Complex number

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- Substitution of 2π in

$$e^{iy} = \cos(y) + i\,\sin(y)$$

$$e^{2\pi i} = \cos(2\pi) + i\,\sin(2\pi)$$

1 0

$$e^{2\pi i} = 1$$

- Substitution of $y = \frac{\pi}{2}, \pi, \frac{-\pi}{2}$ and $-\pi$ will yield

$$e^{\pi i/2} = i, \qquad e^{\pi i} = -1, \qquad e^{-\pi i/2} = -i, \qquad e^{-\pi i} = -1.$$

Another consequence

$$e^{iy} = \cos y + i\sin y.$$

$$\left|e^{iy}\right| = \left|\cos y + i\sin y\right| = \sqrt{\cos^2 y + \sin^2 y} = 1$$

Trigonometric and Hyperbolic Functions.

It can be extended the familiar real trigonometric functions to complex trigonometric functions. We can do this by the use of the Euler formulas.

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x.$$

By addition and subtraction we obtain for the *real* cosine and sine

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}).$$

This suggests the following definitions for complex values $z = x + iy$:

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}). \qquad \cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}),$$

Furthermore

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$$

Derivatives

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z \qquad \frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z$$

EXAMPLE

show that $\cos z = \cos x \cosh y - i \sin x \sinh y$

$$\cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \quad \longrightarrow \quad \cos z = \tfrac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)})$$

$$= \tfrac{1}{2}e^{-y}(\cos x + i \sin x) + \tfrac{1}{2}e^{y}(\cos x - i \sin x)$$

$$= \tfrac{1}{2}(e^{y} + e^{-y}) \cos x - \tfrac{1}{2}i(e^{y} - e^{-y}) \sin x.$$

Since we know

$$\cosh y = \frac{e^{y} + e^{-y}}{2} \qquad \sinh y = \frac{e^{y} - e^{-y}}{2} \quad \text{and}$$

This yield $\cos z = \cos x \cosh y - i \sin x \sinh y$

EXAMPLE

show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$

From above example $\cos z = \cos x \cosh y - i \sin x \sinh y$

and $\cosh^2 y = 1 + \sinh^2 y$ we obtain

$$|\cos z|^2 = (\cos^2 x)(1 + \sinh^2 y) + \sin^2 x \sinh^2 y. \quad \text{Since} \quad \sin^2 x + \cos^2 x = 1,$$

We obtain $|\cos z|^2 = \cos^2 x + \sinh^2 y$

General formula

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2\sin z \cos z \qquad \cos 2z = \cos^2 z - \sin^2 z$$

Hyperbolic Sine and Cosine

For any complex number $z = x + iy$,

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

Additional functions are defined as

$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \coth z = \frac{1}{\tanh z}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} \qquad \operatorname{csch} z = \frac{1}{\sinh z}$$

Similarly we have

$$\frac{d}{dz}\sinh z = \cosh z \quad \text{and} \quad \frac{d}{dz}\cosh z = \sinh z$$

$$\sin z = -i\sinh(iz)\,, \quad \cos z = \cosh(iz)$$

$$\sinh z = -i\sin(iz)\,, \quad \cosh z = \cos(iz)$$

The **natural logarithm** of $z = x + iy$ is denoted by $\ln z$

Then $w = \ln z$ Therefore $e^w = z.$

(Note that $z = 0$ is impossible, since $e^w \neq 0$ for all w; If we set $w = u + iv$ and $z = re^{i\theta}$, this becomes

$$e^w = e^{u+iv} = re^{i\theta}. \qquad z = re^{i\theta}$$

we know that e^{u+iv} has the absolute value e^u and the argument v.

These must be equal to the absolute value and argument on the right:

$$e^u = r, \qquad v = \theta.$$

$e^u = r$ gives $u = \ln r$,

$$\ln z = \ln r + i\theta$$

principal value of ln *z*.

$$\text{Ln } z = \ln |z| + i \text{ Arg } z$$

Example

Find all solutions $z \in \mathbb{C}$ of of the following (express your answers in the form $x+iy$):
(a) $\log z = 4i$;

Solution. (a) We have that $\exp(\log z) = z$. Thus

$$z = \exp(\log z) = \exp(4i) = \cos 4 + i \sin 4$$

General Power

General powers of a complex number $z = x + iy$ are defined by the formula

$$z^c = e^{c \ln z}$$

Since $\ln z$ is infinitely many-valued, z^c will, in general, be multivalued. The particular value

$$z^c = e^{c \operatorname{Ln} z}$$

is called the **principal value** *of* z^c.

If $c = n = 1, 2, \cdots$, then z^n is single-valued and identical with the usual nth power of z. If $c = -1, -2, \cdots$, the situation is similar.

If $c = 1/n$, where $n = 2, 3, \cdots$, then

$$z^c = \sqrt[n]{z} = e^{(1/n) \ln z}$$

Complex integral

Definition of the Complex Line Integral

Complex definite integrals are called (complex) **line integrals**. They are written

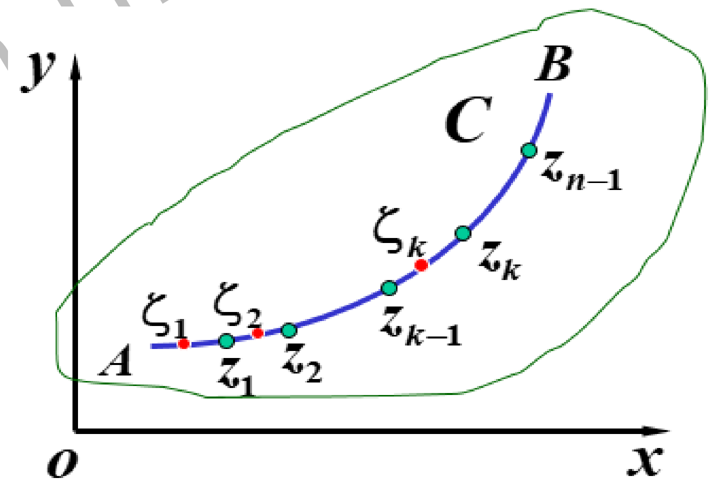
$$\int_C f(z)\, dz.$$

Here the **integrand** is integrated over a given curve C or a portion of it (an *arc*, but we shall say "***curve***" in either case, for simplicity). This curve C in the complex plane is called the **path of integration**.

We assume C to be a **smooth curve**, that is, C has a continuous and differentiable

$$z'(t) = x'(t) + iy'(t)$$

continuous on the entire interval $a \le t \le b$



Geometrically this means that C has everywhere a continuously turning tangent, as follows directly from the definition

$$\dot{z}(t) = \lim_{\Delta t \to 0} \frac{z(t + \Delta t) - z(t)}{\Delta t}$$

Length of C $$L = \int_a^b |z'(t)|\, dt$$

Suppose function $w = f(z)$ is defined in domain D, C is a contour in D from point A to point B. Divide curve C into n segmented lines, the points of division are denoted by

$$A = z_0, z_1, \cdots, z_{k-1}, z_k, \cdots, z_n = B$$

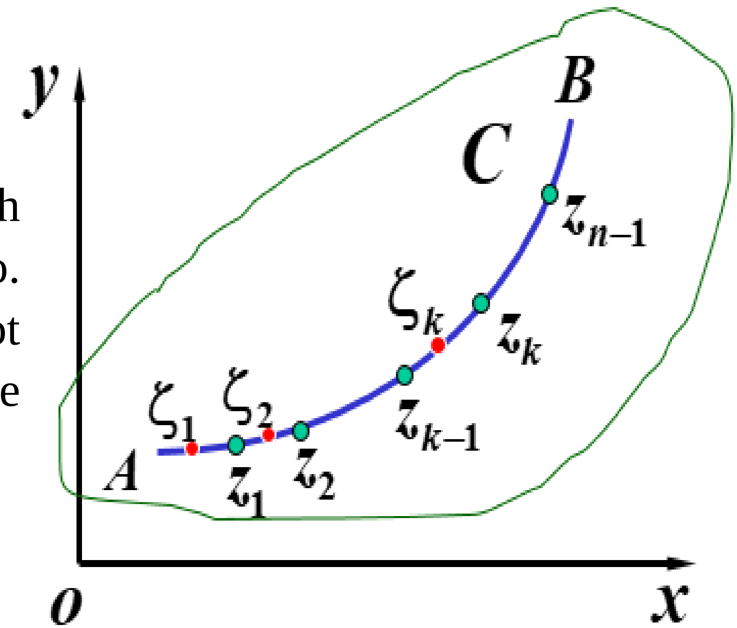
$$z_{k-1} z_k \,(k = 1, 2, \cdots, n)$$

On each portion of subdivision of C we choose an arbitrary point, say, a point ς_1 between z_0 and z_1. On each segment joining z_{k-1} to z_k choose a point ς_k . Form the sum

Let $S_n = \sum_{k=1}^{n} f(\zeta_k)\cdot(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\zeta_k)\cdot\Delta z_k,$

Let Δ be the length of the longest chord Δz_k.

Let the number of subdivisions n approach infinity in such a way that the length of the longest chord approaches zero. The sum Sn will then approach a limit which does not depend on the mode of subdivision and is called the line integral of f(z) from A to B along the curve:



$$\int_C f(z)\mathrm{d}z = \lim_{n\to\infty} \sum_{k=1}^{n} f(\zeta_k)\cdot\Delta z_k.$$

$$\int_a^b f(z)dz = \lim_{\Delta\to 0} \sum_{k=1}^{\infty} f(\xi_k)\Delta z_k$$

First Evaluation Method

Use of a Representation of a Path

This method is not restricted to analytic functions but applies to any continuous complex function

Existence of the Complex Line Integral

Our assumptions that is continuous and C is piecewise smooth imply the existence of the line integral. This can be seen as follows.

$$S_n = \sum (u + iv)(\Delta x_m + i\Delta y_m)$$

$$S_n = \sum (u + iv)(\Delta x_m + i\Delta y_m)$$

where $u = u(\zeta_m, \eta_m)$, $v = v(\zeta_m, \eta_m)$ and we sum over m from 1 to n. Performing the multiplication, we may now split up S_n into four sums:

$$S_n = \sum u\,\Delta x_m - \sum v\,\Delta y_m + i\left[\sum u\,\Delta y_m + \sum v\,\Delta x_m\right].$$

$$\lim_{n\to\infty} S_n = \int_C f(z)\,dz$$

$$= \int_C u\,dx - \int_C v\,dy + i\left[\int_C u\,dy + \int_C v\,dx\right].$$

If **Z(t) = x(t) + i y(t)** for **t** varying between **a** and **b**.

from a point $z = z_1$ to a point $z = z_2$ in the complex plane

$$\int_C f(z)\,dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)\,dz,$$

$$z = z(t) \quad (a \le t \le b)$$

represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$

To compute

$$\int_C f(z)\,dz = \int_a^b f[z(t)]z'(t)\,dt \qquad \dot{z} = \frac{dz}{dt}$$

$$w(t) = u(t) + i v(t)$$

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i \int_a^b v(t)\,dt,$$

$$\text{Re} \int_a^b w(t)\,dt = \int_a^b \text{Re}[w(t)]\,dt \quad \text{and} \quad \text{Im} \int_a^b w(t)\,dt = \int_a^b \text{Im}[w(t)]\,dt.$$

$$\int_C f(z)\,dz = \int_a^b f(z)\,z'(t)\,dt$$

The integral of *f(z)* along the contour C is denoted as follows:

$$\int_a^b f[z(t)]\dot{z}(t)\,dt = \int_a^b (u + iv)(\dot{x} + i\dot{y})\,dt$$

$$= \int_C [u\,dx - v\,dy + i(u\,dy + v\,dx)]$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx).$$

Theorem 1

Integration by the Use of the Path

Let C be a piecewise smooth path, represented by $z = z(t)$, where $a \leqq t \leqq b$. Let $f(z)$ be a continuous function on C. Then

(10) $$\int_C f(z)\,dz = \int_a^b f[z(t)]\dot{z}(t)\,dt \qquad \left(\dot{z} = \frac{dz}{dt}\right).$$

Basic Properties Directly Implied by the Definition

1.Linearity

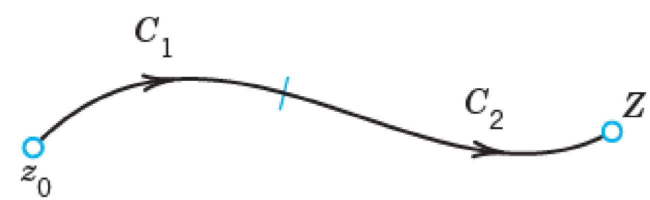
$$\int_C [k_1 f_1(z) + k_2 f_2(z)]\, dz = k_1 \int_C f_1(z)\, dz + k_2 \int_C f_2(z)\, dz.$$

2. Sense reversal in integrating over the ***same*** path, from z_0 to Z (left) and from Z to z_0 (right), introduces a minus sign as shown

$$\int_{z_0}^{Z} f(z)\, dz = -\int_{Z}^{z_0} f(z)\, dz.$$

3. Partitioning of path

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz.$$



Further properties

If C is a closed curve, then the integral on this curve is expressed as

$$\oint_C f(z)dz$$

Steps of solution

(A) Represent the path C in the form $z(t)$ $(a \leqq t \leqq b)$.

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y).

(D) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

A general form we can use for straight lines is

$$z(t) = (1-t)z_1 + tz_2 \text{ with } 0 \leq t \leq 1$$

***Example1**:*

Evaluate $\int_C z\,dz$, C: straight-line segment from point 0 to point $3+4i$.

***Solution**:*

The line equation is $\begin{cases} x = 3t, \\ y = 4t, \end{cases}$ $0 \leq t \leq 1$,

in C, $z = (3+4i)t$, $dz = (3+4i)dt$,

$$\int_C z\,dz = \int_0^1 (3+4i)^2 t\,dt = (3+4i)^2 \int_0^1 t\,dt$$

$$= \frac{(3+4i)^2}{2}.$$

because $\int_C z\,dz = \int_C (x+iy)(dx+idy)$

then regardless of the curve movement to point $3+4i$

$$\int_C z\,dz = \frac{(3+4i)^2}{2}.$$

(A) Represent the path C in the form $z(t)$ $(a \leqq t \leqq b)$.

(B) Calculate the derivative $\dot{z}(t) = dz/dt$.

(C) Substitute $z(t)$ for every z in $f(z)$ (hence $x(t)$ for x and $y(t)$ for y).

(D) Integrate $f[z(t)]\dot{z}(t)$ over t from a to b.

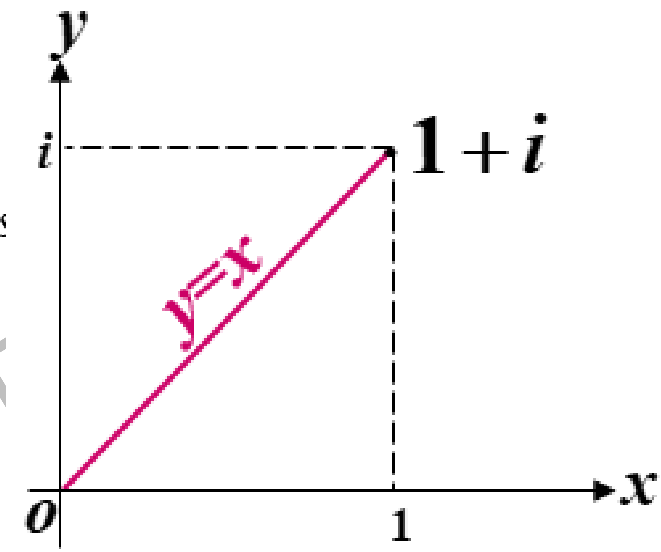
Example

Calculate $\int_C \operatorname{Re} z dz$, where C is:

(1)straight line segment from point zero to point $1+i$;

(2) arc section of parabola $y=x^2$ from point zero to point $1+i$;

(3) polygonal line segment from point zero to point 1 along x axis then from point 1 to point $1+i$.



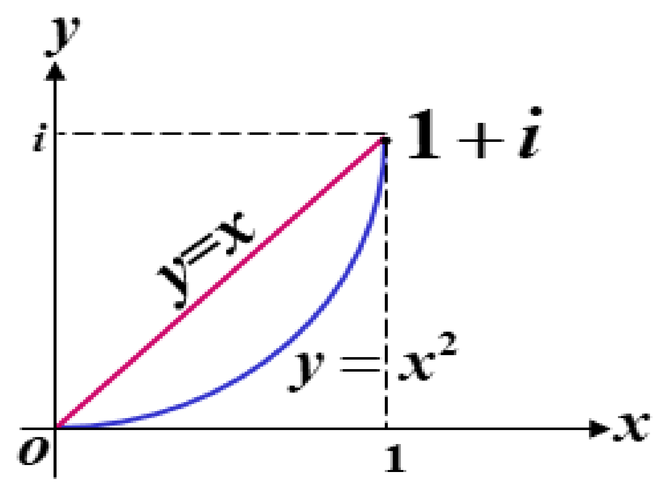
Solution: $z(t)=(1-t)z_1+tz_2$ with $0\le t\le 1$

$$z(t)=t+it \quad (0\le t\le 1),$$

hence $\operatorname{Re} z=t, \quad dz=(1+i)dt,$

$$\int_C \operatorname{Re} z dz = \int_0^1 t(1+i)dt = \frac{1}{2}(1+i);$$

(2) parametric equation is



$$z(t)=t+it^2 \quad (0\le t\le 1),$$

hence $\operatorname{Re} z=t, \quad dz=(1+2ti)dt,$

$$\int_C \operatorname{Re} z dz = \int_0^1 t(1+2it)dt$$

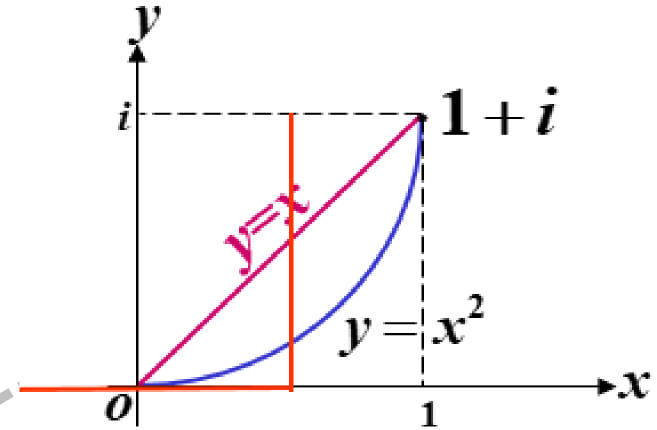
$$=\left(\frac{t^2}{2}+\frac{2i}{3}t^3\right)\Big|_0^1=\frac{1}{2}+\frac{2}{3}i;$$

(3) integration path is composed by two line segments parametric equation of straight-line segment along x axis is

$z(t) = t \quad (0 \le t \le 1),$ hence $\operatorname{Re} z = t, \quad dz = dt,$

parametric equation of straight-line segment from point 1 to point $1+i$ is $z(t) = 1 + it \ (0 \le t \le 1),$

hence $\operatorname{Re} z = 1, \quad dz = i\,dt,$



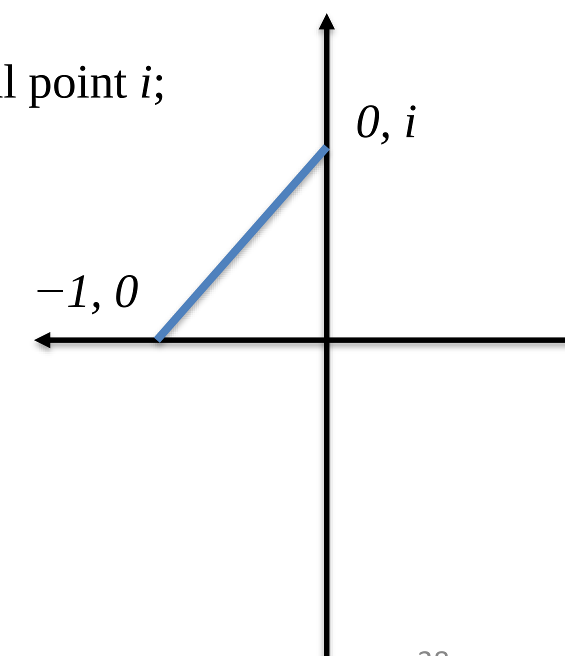
$$\int_C \operatorname{Re} z\,dz = \int_0^1 t\,dt + \int_0^1 1 \cdot i\,dt = \frac{1}{2} + i.$$

$$z(t) = (1-t)z_1 + tz_2 \text{ with } 0 \le t \le 1$$

Example

Evaluate the integral $\int_C |z|^2\,dz$

where the contour C is the line segment with initial point -1 and final point i;



Solution $z(t) = (1-t)z_1 + tz_2$ with $0 \le t \le 1$

(a) Parameterize the line segment by

$$z = -1 + (1+i)t, \qquad 0 \le t \le 1,$$

So that

$$|z|^2 = (-1+t)^2 + t^2 \qquad \text{and} \qquad dz = (1+i)\,dt.$$

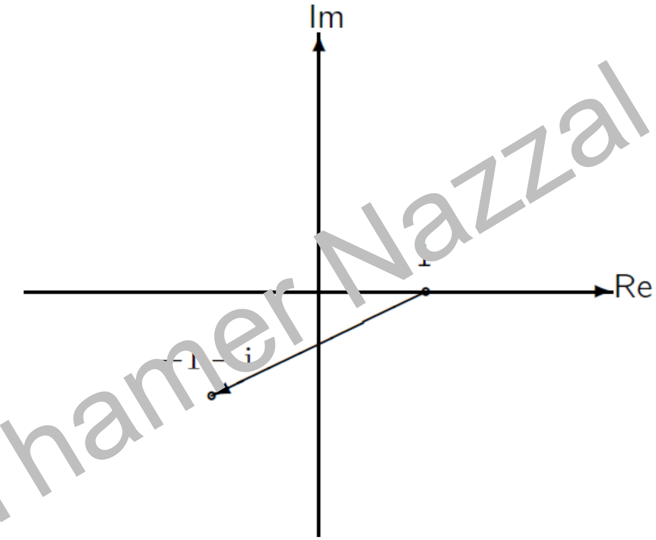
The value of the integral becomes

$$\int_C |z|^2\,dz = \int_0^1 (2t^2 - 2t + 1)(1+i)\,dt = \frac{2}{3}(1+i).$$

Example

Let C be the line from $z_1 = 1$ to $z_2 = -1 - \mathrm{i}$.

Calculate $$I = \int_C |z|^2 \, \mathrm{d}z.$$



Solutio

Recall $z(t) = (1-t)z_1 + tz_2$ with $0 \le t \le 1$

$$z(t) = (1-t)1 + t(-1-\mathrm{i}) = 1 - 2t - \mathrm{i}t \quad \text{with} \quad 0 \le t \le 1$$

Substitute $z(t)$ into the integrand. Note: $\frac{dz}{dt} = (-2-\mathrm{i})$ so multiply the integrand by $(-2-\mathrm{i})$ to get

$$I = \int_0^1 |1 - 2t - \mathrm{i}t|^2 (-2-\mathrm{i}) \, \mathrm{d}t.$$

$(-2-\mathrm{i})$ is a constant and can be taking out of the integral, then further simplification yields

$$I = (-2-\mathrm{i}) \int_0^1 ((1-2t)^2 + t^2) \, \mathrm{d}t = -(2+\mathrm{i}) \int_0^1 (1 - 4t + 5t^2) \, \mathrm{d}t.$$

Now you can integrate the function to obtain

$$I = (-2-\mathrm{i}) \left[t - 2t^2 + \frac{5}{3}t^3 \right]_0^1 = -\frac{4}{3} - \frac{2\mathrm{i}}{3},$$

and so the final result we get is

$$I = \int_C |z|^2 \, \mathrm{d}t = -\frac{4}{3} - \frac{2\mathrm{i}}{3}.$$

Second Evaluation Method: Indefinite Integration and Substitution of Limits

This method is the analog of the evaluation of definite integrals in calculus by the well-known Formula

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

It is suitable for analytic functions only.

Theorem 2

Indefinite Integration of Analytic Functions

Let $f(z)$ be analytic in a simply connected domain D. Then there exists an indefinite integral of $f(z)$ in the domain D, that is, an analytic function $F(z)$ such that $F'(z) = f(z)$ in D, and for all paths in D joining two points z_0 and z_1 in D we have

(9)
$$\int_{z_0}^{z_1} f(z)\,dz = F(z_1) - F(z_0) \qquad [F'(z) = f(z)].$$

(Note that we can write z_0 and z_1 instead of C, since we get the same value for all those C from z_0 to z_1.)

EXAMPLE 1

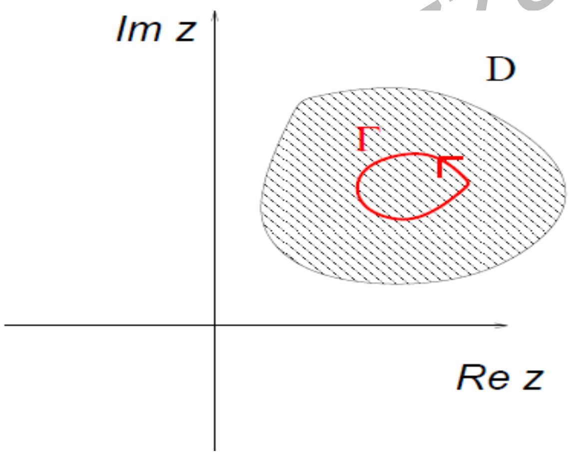
$$\int_0^{1+i} z^2\,dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(1+i)^3 = -\frac{2}{3} + \frac{2}{3}i$$

EXAMPLE 2

$$\int_{-\pi i}^{\pi i} \cos z\,dz = \sin z\Big|_{-\pi i}^{\pi i} = 2\sin \pi i = 2i\sinh \pi = 23.097i$$

Cauchy's integral theorem: If $f(z)$ is analytic in a simply connected region R, [and $f'(z)$ is continuous throughout this region,] then for any closed path C in R, the contour integral of $f(z)$ around C is zero:

$$\oint_C f(z)dz = 0$$



- First, note that if $f(z) = w = u + iv$, then

$$\oint_C f(z)\,dz = \oint_C u\,dx - v\,dy + i\oint_C v\,dx + u\,dy\,;$$

then use Stokes's theorem (see below).

- Construct 2D vectors $\underline{A} = u\hat{\underline{x}} - v\hat{\underline{y}},\quad \underline{B} = v\hat{\underline{x}} + u\hat{\underline{y}},\quad d\underline{r} = dx\,\hat{\underline{x}} + dy\,\hat{\underline{y}}$ in the $xy-$plane and write the integral above as

$$\oint_C f(z)\,dz = \oint_C \underline{A}\cdot d\underline{r} + i\oint_C \underline{B}\cdot d\underline{r} \overset{\text{Stokes's Theorem}}{=} \int_{\text{interior of } C} (\nabla\times\underline{A})\cdot\hat{\underline{z}}\,dS + i\int_{\text{interior of } C} (\nabla\times\underline{B})\cdot\hat{\underline{z}}\,dS\,,$$ but

$$\hat{\underline{z}}\cdot(\nabla\times\underline{A}) = \hat{\underline{z}}\cdot\begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & -v & 0 \end{vmatrix} = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \overset{\text{C.R. cond's}}{=} 0,\qquad \hat{\underline{z}}\cdot(\nabla\times\underline{B}) = \hat{\underline{z}}\cdot\begin{vmatrix} \hat{\underline{x}} & \hat{\underline{y}} & \hat{\underline{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v & u & 0 \end{vmatrix} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \overset{\text{C.R. cond's}}{=} 0$$

$$\Rightarrow \boxed{\oint_C f(z)\,dz = 0}$$